

Collective modes and electromagnetic response of a chiral superconductor

Rahul Roy and Catherine Kallin

*Department of Physics and Astronomy, McMaster University
Hamilton, Ontario, Canada L8S 4M1*

Motivated by the recent controversy surrounding the Kerr effect measurements in strontium ruthenate [1], we examine the electromagnetic response of a clean chiral p-wave superconductor. When the contributions of the collective modes are accounted for, the Hall response in a clean chiral superconductor is smaller by several orders of magnitude than previous theoretical predictions and is too small to explain the experiment. We also uncover some unusual features of the collective modes of a chiral superconductor, namely, that they are not purely longitudinal and couple to external transverse fields.

PACS numbers:

Recent optical experiments by Xia *et al.* [1] on the polar Kerr effect in strontium ruthenate, Sr_2RuO_4 , have been interpreted as evidence for broken time reversal symmetry in the superconducting state. Early indications of broken time reversal symmetry in superconducting Sr_2RuO_4 came from muon spin resonance experiments [2]. These results, together with crystal symmetry and energetic considerations [3], point to a chiral p-wave superconducting order analogous to the superfluid order of the $^3\text{He-A}$ phase [4, 5]. Recent Josephson interferometry measurements [6] have also been interpreted as evidence for chiral p-wave order in superconducting Sr_2RuO_4 . Such a superconducting state, if confirmed, is expected to have many exciting implications for exotic physics [7].

The experiment by Xia *et al.* found a Kerr angle of approximately 60 nanoradians at a frequency, $\omega \simeq 0.8\text{eV}$, which is small compared to the Kerr angle observed in typical ferromagnets, but can be understood qualitatively if one notes that the superconducting gap (or order parameter) for Sr_2RuO_4 is substantially reduced from that of a typical ferromagnet [1]. The Kerr angle at high frequencies is related to the ac Hall conductivity. Theoretical work, however, is divided on the issue of whether the experimental observation result is consistent with the linear response theory of a clean chiral p-wave superconductor [8, 9, 10]. Earlier works on the electromagnetic response of a chiral p-wave superconductor, with an isotropic Fermi surface [10, 11], found no quasi-particle contribution in the clean limit, even in the presence of particle-hole asymmetry, but did predict a Kerr angle due to the so-called “flapping” collective mode. The predicted magnitude is smaller by several orders of magnitude than that observed in Sr_2RuO_4 .

These earlier theoretical studies neglected the effect of an anomalous density-current correlation function which vanishes for a non-chiral superconductor. It was recently argued that when this anomalous correlation function is taken into account, the linear response theory does predict an ac Hall conductivity which is large enough to account for the experiments [8, 9]. In the effective action

language, the anomalous correlation function gives rise to a Chern-Simons-like term which is reminiscent of the quantum Hall effect [12]. A similar term in the Ginzburg Landau free energy leads to a small “spontaneous Hall effect” in a finite system [13]. When the dynamics of the spin degrees of freedom which are unimportant for the purposes of the present paper are also considered, the effective action of a chiral superconductor or superfluid also contains a non-abelian Chern Simons term which is responsible for the spin quantum Hall effect in these systems [14, 15, 16, 17].

In this paper, we examine afresh the linear response of a chiral p-wave superconductor taking into account the anomalous density-current correlation functions and also the contributions from collective modes. We find that when both factors are taken into account, the ac Hall response is strictly zero for an idealized beam normally incident on the a-b plane, with no in-plane wave vector. This is in contrast to the recent results which also considered the effects of the anomalous correlation function [8, 9]. As will be shown below, the discrepancy can be attributed to the contribution from a term in the supercurrent response which was assumed to be negligible at high frequencies [9] but which, in fact, exactly cancels the effect found for zero in-plane wave vector. Nevertheless, for EM waves with a small non-zero in plane wave vector, the ac Hall conductivity is small but non-zero as has been previously shown in two-dimensions [18]. A byproduct of our investigations is the uncovering of some unusual features of the collective modes such as a novel coupling between the collective modes and transverse EM waves and a transverse current associated with these modes.

A chiral superconductor has many properties that are markedly different from a non-chiral superconductor. The chirality of a $p+ip$ superfluid leads to the presence of edge states which in turn give rise to a macroscopic edge current in a neutral superfluid [19, 20]. In a superconductor, these edge currents are screened due to the Meissner effect; nevertheless, they are predicted to be substantial. In both cases, the presence of these edge currents can be traced to the presence of the Chern-Simons-like term in

the effective action and in the Ginzburg Landau expression for the free energy [13, 19]. These edge currents in a neutral chiral superfluid, such as a droplet of superfluid ^3He in the A phase, contribute a macroscopic angular momentum which is proportional to the number of particles in the system [20, 21]. The failure of experiments to detect these edge states [22, 23] as well as other signatures of chirality in strontium ruthenate has led to doubts about the validity of the proposed chiral order parameter structure [5]. Therefore, it is of considerable interest to determine whether the optical Kerr measurements on strontium ruthenate can be explained within the theory of chiral p-wave superconductivity.

We begin by using the low energy effective action of a two dimensional chiral superconductor to study its linear response in Section I. Using the continuity equation, we show that, in contrast to a non-chiral superconductor, the collective modes are excited by a transverse electromagnetic wave. We then study linear response in a three dimensional theory in Sec. II and obtain expressions for the Hall conductivity in Sec. III. Finally, we discuss the polar Kerr effect and compare our calculations with experiments in section IV.

I. LINEAR RESPONSE AND EFFECTIVE ACTION IN A 2D MODEL

The linear response of a planar chiral superconductor can be studied using the low energy phase only effective action which is obtained by integrating out the fermionic degrees of freedom from the theory :

$$e^{iS_{\text{eff}}} = \int d[\psi]d[\psi^\dagger] \exp\left(\frac{i}{2} \int d^2x dt (\psi^\dagger(\partial_t - H)\psi)\right), \quad (1)$$

where ψ, ψ^\dagger are the Grassmanian fields in the two component Nambu formalism and H is the Bogoliubov de Gennes Hamiltonian. For a 2D chiral superconductor, this effective action can be written as follows [18, 19, 24]:

$$\begin{aligned} S_{\text{eff}}(A, \Phi) = & - \int d^2x dt \left[\rho_s \left(\frac{\partial\Phi/2}{\partial t} + eA_0 \right) \right. \\ & + \frac{\rho_s}{2m} \left\{ (\nabla\Phi/2 - e\mathbf{A})^2 - \frac{1}{c_s^2} \left(\frac{\partial\Phi/2}{\partial t} + eA_0 \right)^2 \right\} \\ & \left. + c_{xy} \left(\frac{1}{e} \frac{\partial\Phi/2}{\partial t} + A_0 \right) (\nabla \times \mathbf{A})_z \right], \quad (2) \end{aligned}$$

where ρ_s is the equilibrium superfluid density, $\Phi(\mathbf{r}, t)$ is the phase of the superconducting order parameter and c_s is the speed of sound. The first two terms are the only terms present for an s-wave superconductor in the London limit of constant superfluid density, ρ_s , and the third term arises from fluctuations of the superfluid density. The above effective action is applicable for a single sheet of a chiral superconductor. The effective action for n decoupled sheets is obtained by multiplying

the above action by n . The term containing c_{xy} is the Chern-Simons-like term which, as pointed out above, arises from the density-current correlation function. In general, c_{xy} is frequency dependent as discussed below. The c_{xy} term is not the full Chern Simons term, because it is missing the term $\epsilon_{0ij} \mathbf{A}_i \partial_t \mathbf{A}_j$ [18, 19]. The action is nevertheless invariant under gauge transformations: $\Phi \rightarrow \Phi + 2\theta, \mathbf{A} \rightarrow \mathbf{A} + \frac{1}{e} \nabla\theta, A_0 \rightarrow A_0 - \frac{1}{e} \partial_t \theta$.

The current density response is obtained in the usual manner:

$$\mathbf{j} = \frac{\delta S}{\delta \mathbf{A}} = \frac{e\rho_s}{m} (\nabla\Phi/2 - e\mathbf{A}) + \mathbf{j}^{\text{cs}}, \quad (3)$$

where the anomalous part of the current density, arising from the Chern-Simons-like term, is

$$\mathbf{j}^{\text{cs}} = c_{xy} \hat{\mathbf{z}} \times \left[\nabla A_0 + \frac{\partial(\nabla\Phi/2)}{e\partial t} \right]. \quad (4)$$

Similarly, the charge density is:

$$\rho = -\frac{\delta S_{\text{eff}}}{\delta A_0} = e\rho_s - \frac{e\rho_s}{mc_s^2} \left(\frac{\partial\Phi/2}{\partial t} + eA_0 \right) + c_{xy} (\nabla \times \mathbf{A})_z. \quad (5)$$

We note that the anomalous current density can be rewritten in the following form:

$$\mathbf{j}^{\text{cs}} = c_{xy} \hat{\mathbf{z}} \times \left[-\mathbf{E} + \frac{\partial(\nabla\Phi/2 - e\mathbf{A})}{e\partial t} \right]. \quad (6)$$

At high frequencies, in the long wavelength limit, the expressions for the anomalous current response is obtained by simply replacing the coefficient c_{xy} by a frequency dependent one $c_{xy}(\omega)$ as obtained from the form of the correlation function.

It was argued that at high frequencies, the second term in Eq. (6) can be neglected, giving rise to an anomalous current which flows perpendicular to the electric field [8, 9]. The expression for the Hall conductivity thus obtained agrees quite well with the value extracted from the observed optical Kerr effect. However, in a non-chiral superconductor, the second term is proportional to the time-derivative of the superfluid current, which itself is proportional to the electric field, $\partial\mathbf{j}/\partial t = \rho_s e^2 \mathbf{E}/m$, so that the two terms exactly cancel, giving a vanishing Hall conductivity. This is also the case for a clean chiral superconductor, since this relation between the current and the electric field follows from translational symmetry, and must hold independent of any interactions which lead to superconductivity. In the clean limit, the ac conductivity is simply $\sigma_{\alpha,\beta}(k=0, \omega) = \delta_{\alpha,\beta} \rho_s e^2 / (m\omega + i\epsilon)$, so that the bulk ac Hall conductivity, $\sigma_{xy}(k=0, \omega)$, vanishes at all frequencies [16]. It is possible to have a small spontaneous dc Hall conductivity in a finite sample, due to edge effects [13]. Furthermore, both the Chern-Simons-like term, and contributions from the flapping collective modes can give rise to a spontaneous ac Hall

response at finite wavevector, even in the ideal, clean limit [10, 18, 25].

The finite wave vector response can be studied by integrating out the superconducting phase from the effective action, to obtain an expression for the current response as a function of the total field [18, 25]. Here, we adopt a quantum hydrodynamic approach by starting from the above effective action and using the continuity equation to determine the dynamics of the collective modes. This approach makes the contributions from the collective modes and their coupling to the electromagnetic waves transparent. Approaches which are somewhat similar have also appeared in the literature [26, 27].

The equation of motion for the phase of the order parameter is simply the continuity equation:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \quad (7)$$

In this section, we consider a two-dimensional model (with no dependence on z , so that in Fourier space, $k_z=0$) and work in the transverse gauge, $\nabla \cdot \mathbf{A}=0$, for simplicity. In this case, the above continuity equation yields:

$$\frac{e\rho_s}{m} \nabla^2(\Phi/2) - \frac{e\rho_s}{mc_s^2} \left(\frac{\partial^2 \Phi/2}{\partial t^2} - e \frac{\partial A_0}{\partial t} \right) + c_{xy} \frac{\partial B_z}{\partial t} = 0. \quad (8)$$

It follows that, in the transverse gauge, the phase is decoupled from the field except when at least one of $\frac{\partial B_z}{\partial t}$ or $\frac{\partial A_0}{\partial t}$ is nonzero. The coupling of the phase variable to a transverse field is a novel feature of a chiral superconductor. Unlike a conventional superconductor where such a coupling may arise due to mass anisotropy [28], in a chiral superconductor, this coupling persists even when the Hamiltonian has Galilean invariance [35]. Due to this coupling, an external transverse EM wave can give rise to charge density oscillations which generate a scalar Coulomb potential A_0 . The dispersion of these modes will be examined in the next section. There is an appealing physical picture for this coupling. It was noted in Ref. [19] that the coupling of the density to the z -component of the magnetic field in Eq. (5) can be understood as arising from the diamagnetic coupling of the Cooper pairs which have an intrinsic magnetic moment (due to the nonzero angular momentum), with the external magnetic field. In this case, when the magnetic field oscillates in time, this coupling gives rise to density oscillations and hence excites the collective modes. In a charged superfluid, these density oscillations give rise to an internal field described by the scalar Coulomb potential in Eq. (8).

In linear response, one can simply take the Fourier transform of the above equation to obtain

$$\frac{e\rho_s}{mc_s^2} (\omega^2 - c_s^2 k^2) \Phi(\mathbf{k}, \omega)/2 = ic_{xy} \omega B_z - \frac{e^2 \rho_s}{mc_s^2} i \omega A_0, \quad (9)$$

where \mathbf{k} is a 2d vector. Thus the collective phase field excited by the magnetic field is given by

$$\Phi(\mathbf{k}, \omega)/2 = \frac{i\omega (mc_s^2 c_{xy} B_z - e^2 \rho_s A_0)}{e\rho_s (\omega^2 - c_s^2 k^2)}. \quad (10)$$

Putting this back into the equation for the current response, we see that the collective mode contributes in *two* ways. It enters into the usual term, as well as into the anomalous or the Chern-Simons-like term.

We write the current as a sum of two terms, $\mathbf{j} = \mathbf{j}_d + \mathbf{j}_\Phi$, where \mathbf{j}_d is the “direct” response and is given by

$$\mathbf{j}_d = \frac{-e^2 \rho_s}{m} (\mathbf{A}) + c_{xy} (\hat{\mathbf{z}} \times i \mathbf{k} A_0). \quad (11)$$

This is the only term which exists when the collective modes are not excited, i.e., when $dB_z/dt = 0$ and $dA_0/dt = 0$, and is the only term to contribute at $k = 0$. The other term, \mathbf{j}_Φ , is the current response due to the collective modes and is given by

$$\mathbf{j}_\Phi = \frac{e\rho_s}{m} (\nabla \Phi/2) + c_{xy} \hat{\mathbf{z}} \times \frac{\partial (\nabla \Phi/2)}{e \partial t} \quad (12)$$

$$= \left[\left(\frac{e\rho_s}{m} \right) i \mathbf{k} + \frac{c_{xy} \omega}{e} \hat{\mathbf{z}} \times \mathbf{k} \right] \Phi(\mathbf{k}, \omega)/2. \quad (13)$$

The Hall current \mathbf{j}_H , comes from terms in the current response that are linear in c_{xy} :

$$\mathbf{j}_H = \frac{c_{xy} c_s^2}{(\omega^2 - c_s^2 k^2)} (k^2 \hat{\mathbf{z}} \times (-i \mathbf{k} A_0) - \mathbf{k} \omega B_z). \quad (14)$$

Rewriting $-i \mathbf{k} A_0$ as $\mathbf{E} - i \omega \mathbf{A}$ and noting that $B_z = -i(\hat{\mathbf{z}} \times \mathbf{A}) \cdot \mathbf{k}$ and $\mathbf{k} \cdot \mathbf{A} = 0$, the expression for the Hall current reduces to

$$\mathbf{j}_H = c_{xy} \frac{c_s^2 k^2 (\hat{\mathbf{z}} \times \mathbf{E})}{\omega^2 - c_s^2 k^2}. \quad (15)$$

Thus the Hall conductivity, defined in this geometry as $\sigma_{xy} = j_x/E_y$, is:

$$\sigma_{xy} = c_{xy} \frac{-c_s^2 k^2}{\omega^2 - c_s^2 k^2}. \quad (16)$$

We have thus recovered the result of Refs. [18, 25] that the Hall conductivity has a k^2 dependence and vanishes in the limit $k \rightarrow 0$ at finite ω , as required by Galilean invariance. The above analysis was restricted to EM waves propagating in the plane of a two dimensional material. A more general analysis is presented in the next section.

II. 3D LAYERED MODEL AND COLLECTIVE MODES

In order to connect to experiments on Sr_2RuO_4 , we need to go beyond the two-dimensional model considered above. Sr_2RuO_4 is extremely anisotropic, but with

coherent transport along the c-axis [29]. The low energy effective action describing such a layered superconductor can be written, in analogy with Eq. (2), as:

$$S_{\text{eff}}(A, \Phi) = - \int d^3x dt \left[\rho_s \left(\frac{\partial \Phi/2}{\partial t} + eA_0 \right) + \sum_i \frac{\rho_s}{2m_i} (\partial_i \Phi/2 - eA_i)^2 - \frac{\rho_s}{2mc_s^2} \left(\frac{\partial \Phi/2}{\partial t} + eA_0 \right)^2 + \tilde{c}_{xy} \tilde{z} \left(\frac{1}{e} \frac{\partial \Phi/2}{\partial t} + A_0 \right) (\nabla \times \mathbf{A})_z \right], \quad (17)$$

where ρ_s now represents the superfluid density in three dimensions and the mass parameters m_i have been used to represent the anisotropic diamagnetic current response in the long wavelength limit. We restrict ourselves to the case where $m_x = m_y = m$. In the two dimensional limit, $m_z \rightarrow \infty$, the Chern Simons coefficient $c_{xy\tilde{z}}$ becomes $c_{xy\tilde{z}} = \frac{c_{xy}}{a}$, where a is the interlayer separation. More generally, we can write $c_{xy\tilde{z}} = \alpha c_{xy}$ where α is a parameter which has the dimensions of inverse length. Hereafter, we use the notation \tilde{c}_{xy} . It should also be noted that the velocity of sound, c_s , is no longer that applicable for the two dimensional case and depends on the details of the microscopic Hamiltonian. Here, we do not specialize to a particular Hamiltonian.

The current density is then given by

$$\mathbf{j} = -\frac{e^2 \rho_s}{m} \tilde{\mathbf{A}} + \tilde{c}_{xy} (\hat{\mathbf{z}} \times \nabla A_0) + \frac{e \rho_s}{m} i \tilde{\mathbf{k}} (\Phi/2) + \frac{\tilde{c}_{xy} \omega}{e} (\hat{\mathbf{z}} \times \mathbf{k}) (\Phi/2), \quad (18)$$

where $\tilde{\mathbf{k}} = (k_x, k_y, k_z \frac{m}{m_z})$ and $\tilde{\mathbf{A}} = (A_x, A_y, A_z \frac{m}{m_z})$. The expression for the charge density has the same form as Eq. (5) with the parameter c_{xy} replaced by \tilde{c}_{xy} and ρ_s by the appropriate three dimensional superfluid density.

The collective mode response can again be determined using the continuity equation, as in the two dimensional case, and gives:

$$\Phi/2(\mathbf{k}, \omega) = \frac{\tilde{c}_{xy} i \omega B_z + \frac{e^2 \rho_s}{m} i \tilde{\mathbf{k}} \cdot \mathbf{A} - \frac{e^2 \rho_s}{mc_s^2} i \omega A_0}{\frac{e \rho_s}{mc_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})}. \quad (19)$$

In calculating the electromagnetic response, the Coulomb interaction does not appear explicitly because all fields and gauge potentials correspond to the total fields. However, to obtain the dispersion relation for the collective modes, one needs to include the effect of Coulomb interactions. This is most simply done by using the self consistent field implementation of the random phase approximation [30]. We replace A_0 by $\frac{4\pi}{\epsilon k^2}$ (where ϵ is the dielectric constant of the system that comes from sources other than the conduction electrons) in Eq. (18),

set all other fields to zero and use the continuity equation. On doing this we obtain :

$$\left[\frac{e \rho_s \omega^2}{mc_s^2} \left(1 + \frac{\omega_p^2}{c_s^2 k^2} \right)^{-1} - \frac{e \rho_s}{m} \mathbf{k} \cdot \tilde{\mathbf{k}} \right] \frac{\Phi}{2} = 0. \quad (20)$$

This gives the dispersion of the collective modes to be

$$\omega^2 = c_s^2 \left(1 + \frac{\omega_p^2}{\epsilon c_s^2 k^2} \right) \mathbf{k} \cdot \tilde{\mathbf{k}}, \quad (21)$$

where $\omega_p^2 = \frac{4\pi \rho e^2}{m}$. The dispersion is identical to that of an anisotropic s-wave superconductor, where the anisotropy enters through $\tilde{\mathbf{k}}$. The long-wavelength form of the dispersion arises due to the long range nature of the Coulomb interaction and is thus independent of whether the superconductor is chiral or not. In analogy with the usual definition of the plasma frequency, we can define two different plasma frequencies corresponding to oscillations in the a-b plane, ω_{ab}^p for $\mathbf{k} = (k_x, k_y, 0)$ and along the z-direction, ω_c^p for $\mathbf{k} = (0, 0, k_z)$:

$$\omega_{ab}^p = \frac{4\pi \rho e^2}{m}; \quad \omega_c^p = \frac{4\pi \rho e^2}{m_z}. \quad (22)$$

It follows from Eq.(18), that the current of the collective mode for a superconductor has a transverse component $\frac{\tilde{c}_{xy} \omega}{e} (\hat{\mathbf{z}} \times \mathbf{k}) (\Phi/2) + \tilde{c}_{xy} (\hat{\mathbf{z}} \times \nabla A_0)$ [36]. This is in addition to any transverse components which may arise from mass anisotropy. To differentiate this mixing of longitudinal and transverse degrees of freedom from that which arises purely due to mass anisotropy, one can consider the 2D limit $m_z \rightarrow \infty$ for \mathbf{k} vectors lying in the x-y plane. It is clear that a chiral 2 d superconductor, even one with Galilean invariance [37] will have collective modes which have a transverse component. Due to the chiral nature of the order, the collective modes are no longer purely longitudinal.

III. RESPONSE OF A CHIRAL 3D SUPERCONDUCTOR

The current in Eq. (18) can be written as

$$\mathbf{j} = \mathbf{j}^0 + \mathbf{j}^{cs}, \quad (23)$$

where \mathbf{j}^{cs} contains all the terms proportional to \tilde{c}_{xy} and no other terms and \mathbf{j}^0 is the current density for $\tilde{c}_{xy} = 0$. For the remainder of the paper, our focus shall be on the anomalous part of the current.

From Eqs. (18) and (19), the anomalous part of the current response can be written in terms of the external electric and magnetic fields as

$$\mathbf{j}^{cs} = \frac{\tilde{c}_{xy} (\hat{\mathbf{z}} \times \mathbf{k}) c_s^2 (\tilde{\mathbf{k}} \cdot \mathbf{E})}{(\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})} + \frac{i c_s^2 \tilde{\mathbf{k}} \tilde{c}_{xy} i \omega B_z}{(\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})} + \frac{(\tilde{c}_{xy})^2 \omega (\hat{\mathbf{z}} \times \mathbf{k}) i \omega B_z}{\frac{e \rho_s}{mc_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})}. \quad (24)$$

We can express the current entirely in terms of the electric field, by writing $i\omega B_z = i(\mathbf{k} \times \mathbf{E}) \cdot \hat{\mathbf{z}}$ and, hence, obtain the anomalous part of the conductivity tensor. The anomalous conductivity can be split up into two parts, an antisymmetric part coming from the first term and a symmetric part which comes from the second term :

$$\sigma_{im}^a = \frac{\tilde{c}_{xy} c_s^2}{(\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})} (\epsilon_{3li} k_l \tilde{k}_m - \epsilon_{3lm} \tilde{k}_i k_l) \quad (25)$$

$$\sigma_{im}^s = \frac{\tilde{c}_{xy}^2 c_s^2 \omega m}{e \rho_s (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})} i (\delta_{ln} \delta_{im} - \delta_{lm} \delta_{ni}) k_l k_n. \quad (26)$$

The in-plane Hall conductivity, σ_{xy} is

$$\sigma_{xy} = \frac{-\tilde{c}_{xy} c_s^2 k_{\parallel}^2}{(\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})}, \quad (27)$$

which reduces to the two dimensional expression, Eq. (16) scaled by a factor of α in the limit $m_z \rightarrow \infty$.

The charge density can also be expressed as follows :

$$\delta\rho = -\frac{\tilde{c}_{xy} c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}} B_z}{(\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})} + \frac{e^2 \rho_s}{m} \frac{i \tilde{\mathbf{k}} \cdot \mathbf{E}}{(\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})}, \quad (28)$$

and the internal field generated is

$$\mathbf{E}_{int} = -i A_{int}^0 \mathbf{k} = \frac{-i 4 \pi \delta \rho}{\epsilon k^2} \mathbf{k} \quad (29)$$

$$= \frac{i 4 \pi \tilde{c}_{xy} c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}} B_z \mathbf{k}}{\epsilon k^2 (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})} + \frac{\omega_p^2 (\tilde{\mathbf{k}} \cdot \mathbf{E}) \mathbf{k}}{\epsilon k^2 (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}})}. \quad (30)$$

The vector \mathbf{E} on the right hand side of the equation represents the total electric field. To find the dispersion of the collective modes, we can set $B_z = 0$ and $\mathbf{E} = \mathbf{E}_{int}$ and recover the same dispersion relation that we found previously.

The effective action written above is valid at zero temperature in the low energy limit where $v_f k, \omega \ll \Delta$. At higher frequencies and non zero temperatures, the effective action in momentum space is more conveniently written down in momentum space. The terms $\tilde{c}_{xy}, c_s, \rho$ acquire a frequency and temperature dependence. In addition, there is an extra term in the effective action :

$$S = \int d^3 k d\omega \sum_l c_{l0}(\mathbf{k}, \omega) A_l(-\mathbf{k}, -\omega) A_0(\mathbf{k}, \omega). \quad (31)$$

This term can be neglected in the long wavelength and low frequency limit because it is proportional to both \mathbf{k} and ω in this limit. At high frequencies and long wavelengths, the linear dependence on \mathbf{k} remains, however the frequency dependence changes and this term becomes comparable to other terms in the effective action. This term results in the presence of the following additional terms in the current and density response:

$$\mathbf{j}_l'(\mathbf{k}, \omega) = c_{l0}(\mathbf{k}, \omega) (A_0 - i \frac{\omega}{e} \Phi(\mathbf{k}, \omega)) \quad (32)$$

$$\rho'(\mathbf{k}, \omega) = -c_{l0}(\mathbf{k}, \omega) (A_l - i \frac{k_l}{e} \Phi(\mathbf{k}, \omega)) \quad (33)$$

The response of the collective phase variable becomes :

$$\Phi/2 = \frac{\tilde{c}_{xy} i \omega B_z + \frac{e^2 \rho_s}{m} (i \tilde{\mathbf{k}} \cdot \mathbf{A} - \frac{i \omega A_0}{c_s^2})}{\frac{e \rho_s}{m c_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}}) + 2 \frac{c_{l0} k_l \omega}{e}} - \frac{i c_{l0} (k_l A_0 + A_l \omega)}{\frac{e \rho_s}{m c_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}}) + 2 \frac{c_{l0} k_l \omega}{e}} \quad (34)$$

where a summation over repeated indices is implied and the dependence of $\rho, \tilde{c}_{xy}, c_s$ on ω, T and of c_{l0} on (\mathbf{k}, ω, T) have been suppressed.

The full current and density response and the dispersion of the collective modes also changes and can easily be deduced from Eqs. (24), (28) and (32)-(34). Here we only write down the expressions for the anomalous charge density and the antisymmetric part of the anomalous Hall conductance

$$\delta\rho^{cs} = \tilde{c}_{xy} \frac{B_z (c_{l0} k_l \omega - e^2 \frac{\rho_s}{m} \mathbf{k} \cdot \tilde{\mathbf{k}})}{\frac{e^2 \rho_s}{m c_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}}) + 2 c_{l0} k_l \omega} \quad (35)$$

$$\sigma_{im} = \tilde{c}_{xy} \left[\frac{\epsilon_{3li} k_l (\tilde{k}_m \frac{e^2 \rho_s}{m} - c_{m0} \omega)}{\frac{e^2 \rho_s}{m c_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}}) + 2 c_{l0} k_l \omega} - \frac{\epsilon_{3lm} k_l (\tilde{k}_i \frac{e^2 \rho_s}{m} - c_{i0} \omega)}{\frac{e^2 \rho_s}{m c_s^2} (\omega^2 - c_s^2 \mathbf{k} \cdot \tilde{\mathbf{k}}) + 2 c_{l0} k_l \omega} \right] \quad (36)$$

Using Eq. (36) and the results from the appendix, in the long wavelength and high frequency limit and the 2D limit of $m_z \rightarrow \infty$ we deduce that

$$\sigma_{xy} = \tilde{c}_{xy} \left(-\frac{k_{\parallel}^2 p_f^2}{2 m^2 \omega^2} \right) \quad (37)$$

where p_f is the Fermi momentum.

IV. KERR EFFECT AND DISCUSSION

One experiment which directly probes the anomalous ac Hall conductivity considered here is the polar Kerr effect [1]. In this experiment, linearly polarized light beam which is normally incident on the superconducting planes is reflected back as elliptically polarized light. The polar Kerr angle, which measures the degree of rotation of the polarization, is an indicator of the extent of time reversal symmetry breaking [31, 32]. If the c-axis of strontium ruthenate is chosen to be the z-axis, the Kerr angle is proportional to the real or imaginary part of σ_{xy} depending on whether the real part of the refractive index is much smaller or larger than the complex part (see Appendix).

Connecting the calculations presented here to the Kerr effect experiments reported in Ref. [1] is problematic for two reasons. First, to a very good approximation, the

experiment is done under conditions of light normally incident on the superconducting surface, i.e. $\mathbf{k} = (0, 0, k)$, for which the Hall conductivity calculated above vanishes. Second, the frequency of the incident light, $\omega \simeq 0.8$ eV, is very large compared to the superconducting gap of strontium ruthenate, $\Delta \simeq 0.23$ meV, although probably not large enough to create transitions to higher lying energy bands [33]. While the pairing interaction for strontium ruthenate is not known, this large probing frequency is likely to be beyond the pairing cutoff used in BCS theory. As shown in the Appendix, the pairing cutoff enters the coefficient, c_{xy} , and the ac Hall conductivity is substantially reduced for frequencies above this cutoff. Keeping these two caveats in mind, one can crudely estimate the predicted Kerr angle if one simply assumes the experiment is probing at a frequency below the superconducting pairing cutoff and at a finite in-plane wavevector introduced by the finite size of the laser beam incident on the surface. In this case, taking the complex refractive index to be $1.72i$ [38], one finds using Eqs. (37), (44) and (49) that the Kerr angle is of the order of $\sim 10^{-17}$ radians [39], or roughly nine orders of magnitude smaller than the observed value.

From Eqs. (48) and (49), we see that the Kerr angle is greatly enhanced in the region where $n + i\kappa \sim 0$, which corresponds to $\omega \sim \omega_p/\sqrt{\epsilon}$. In this case, the Kerr angle should be determined using Eq. (47). However, for the predicted Kerr angle to be of the order seen in experiments, the probing frequency would need to be within $\sim 10^{-6}\%$ of $\omega_p/\sqrt{\epsilon}$. While disorder and lifetime effects can be expected to broaden this window, the resonance is sufficiently sharp and the enhancement required sufficiently large that this is very unlikely to explain the experiments.

Disorder can have a substantial effect on the conductivity, since it relaxes the constraints imposed by Galilean invariance. Consequently, disorder-induced terms can contribute significantly to the conductivity tensor at finite frequency and zero wave vector, provided the scattering rate is not too small relative to the superconducting gap. The superconducting transition temperature of Sr_2RuO_4 is very sensitive to disorder and samples exhibiting the maximum T_c of 1.5 K are believed to be in the clean limit. Nevertheless, τ is still estimated to be $\sim 10^{-11}$ s [1] which could substantially alter the results presented here, as well as earlier calculations which investigated some of the quasiparticle and collective-mode contributions to the Kerr effect [10, 11]. However, given the high probing frequency, the concern about being above the pairing frequency cutoff remains.

More generally, our results suggest that it would be most interesting to study the Hall conductivity or Kerr angle of Sr_2RuO_4 at lower frequencies where the signatures of the chiral superconducting order are most pronounced. Of course, it would also be of great interest to directly study the response at finite wave vector, for

example, by superimposing a grating on the sample, although it is always difficult to achieve wave vectors of sufficient size.

Note added: While at KITP in December 2007, we learned that another group, Roman Lutchyn, Pavel Nagornykh and Victor Yakovenko, had achieved similar results which were in substantial agreement with ours, using a somewhat different approach. We thank them for sending us a copy of their manuscript before posting it on the arxiv [34].

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and by the Canadian Institute for Advanced Research. We would like to thank the Kavli Institute for Theoretical Physics, Santa Barbara (supported by the National Science Foundation under Grant No. PHY05-51164) where a part of this work was completed for its hospitality during the mini-program “ Sr_2RuO_4 and Chiral p-wave superconductivity”. We would also like to thank John Berlinsky, R Brout, Aharon Kapitulnik, Roman Lutchyn Vladimir Mineev, Chetan Nayak, Jim Sauls, Michael Stone, and Victor Yakovenko for useful discussions.

APPENDIX

A. High frequency response

The parameters $c_{l0}, \tilde{c}_{xy}, c_s$ at high frequency can be calculated using linear response theory. For simplicity, we consider the $T = 0$ case. The expressions for the finite temperature coefficients can be calculated in an analogous manner [9, 26].

$$\frac{\rho_s}{mc_s^2} = - \int \frac{d^3p}{(2\pi)^3} \frac{4\Delta^2}{\epsilon} \left(\frac{1}{(\omega^2 + i\delta)^2 - (2\epsilon)^2} \right) \quad (38)$$

$$c_{l0} = e^2 \omega k_l \int \frac{d^3p}{(2\pi)^3} \frac{(v_l)^2 \Delta^2}{\epsilon^3} \left(\frac{1}{(\omega^2 + i\delta)^2 - (2\epsilon)^2} \right) \quad (39)$$

$$\tilde{c}_{xy} = -2e^2 \int \frac{d^3p}{(2\pi)^3} \frac{v_x f(p)}{\epsilon} \frac{1}{((\omega^2 + i\delta)^2 - (2\epsilon)^2)} \quad (40)$$

where

$$f(p) = \lim_{q_y \rightarrow 0} \frac{Im \left(\Delta(p_x, p_y + \frac{q_y}{2}) \Delta^\dagger(p_x, p_y - \frac{q_y}{2}) \right)}{q_y}, \quad (41)$$

$\epsilon(p) = \sqrt{\xi(p)^2 + \Delta(p)^2}$ and $\xi(p)$ are the single particle energies of the electronic system and $v_l = \frac{\partial \xi(p)}{\partial p_l}$.

To evaluate these integrals, we take the limit $m_z \rightarrow \infty$ and use the free particle dispersion in 2D: $\xi(p) = p^2/2m - \epsilon_f$ where ϵ_f is the Fermi energy. We take the gap function

to be

$$\Delta(p) = \begin{cases} \Delta_0(p_x + ip_y)/p & \text{if } |\xi(p)| < \omega_D \\ 0 & \text{if } |\xi(p)| > \omega_D \end{cases} \quad (42)$$

where $\omega_D \gg \Delta$ is a BCS cutoff. Then \tilde{c}_{xy} reduces to

$$\tilde{c}_{xy} = \frac{e^2}{8\pi a} \int_{-\omega_D}^{\omega_D} \frac{dx}{\sqrt{1+x^2}(1+x^2-(\omega/2\Delta_0)^2)} \quad (43)$$

$$\tilde{c}_{xy} = \frac{e^2}{4\pi a} \begin{cases} \frac{\sin^{-1}(\alpha)}{\alpha\sqrt{1-\alpha^2}} & \text{for } \omega < 2\Delta \\ \frac{i\pi}{2\alpha^2} - \frac{1}{\alpha^2} \ln(\frac{\omega}{\Delta}) & \text{for } 2\Delta \ll \omega < \omega_D \\ -\frac{1}{\alpha^2} \ln(\frac{\omega_D}{\Delta}) & \text{for } \omega_D \ll \omega \end{cases} \quad (44)$$

where $\alpha = \frac{\omega}{2\Delta_0}$ and a is the interlayer spacing. This reduces in the limit $\omega_D \rightarrow \infty$ to the expressions obtained in Ref. [9] scaled by a factor of $1/a$.

While the BCS cutoff is a crude approximation to the energy dependence of any realistic pairing potential, it is used here to highlight the fact that the effects of the Chern-Simons-like term are only effective close to the Fermi energy. As pointed out by Yakovenko [9], \tilde{c}_{xy} corresponds to the excitation of two BCS quasiparticles (or quasiholes). Such a term typically carries the usual coherence factor, but \tilde{c}_{xy} only carries the piece containing the chiral signature, i.e. $\Delta(p)$, which vanishes at energies above the pairing cutoff.

To obtain the in-plane Hall conductivity σ_{xy} in the long wavelength limit at high frequencies, we also need to calculate

$$\left(\frac{mc_S^2(-(k_x^2 + k_y^2)\frac{e^2\rho_z}{m} + \omega(c_{y0}k_y + c_{x0}k_x))}{e^2\rho_s\omega^2} \right) \quad (45)$$

Using Eqs. 38,39 and 40 ,this quantity reduces at high frequencies in the two dimensional limit of $m_z \rightarrow 0$ to $\left(-\frac{k_{\parallel}^2 p_f^2}{2m^2\omega^2}\right)$ [40].

B. Kerr Angle

Let $n + i\kappa$ be the complex refractive index given by $(n + i\kappa)^2 = \epsilon + \frac{i4\pi\sigma_{xx}}{\omega}$ and $n_{\pm} + i\kappa_{\pm}$ be the complex index of refraction for right and left circularly polarized light. Then [31, 32],

$$(n_{\pm} + i\kappa_{\pm})^2 = 1 + i4\pi\sigma_{\pm}/\omega \quad (46)$$

where $\sigma_{\pm} = \sigma_{xx} \pm i\sigma_{xy}$. The Kerr angle is given by :

$$\theta_K = -\frac{1}{2} \left(-\tan^{-1}\left(\frac{\kappa_+}{1-n_+}\right) + \tan^{-1}\left(\frac{\kappa_-}{1-n_-}\right) - \tan^{-1}\left(\frac{\kappa_+}{1+n_+}\right) + \tan^{-1}\left(\frac{\kappa_-}{1-n_-}\right) \right) \quad (47)$$

When $n \gg \kappa$, then the Kerr angle is given by the formula

$$\theta_K = \frac{4\pi\sigma''_{xy}}{n(n^2-1)\omega} \quad (48)$$

and when $\kappa \gg n$, then

$$\theta_K = \frac{4\pi\sigma'_{xy}}{\omega\kappa^3} \quad (49)$$

In the clean limit, when ω is close to $\omega_p/\sqrt{\epsilon}$, Eq. (47) should be used to calculate the Kerr angle, while far from the resonance, Eqs. (48) and (49) should be used when $\omega > \omega_p/\sqrt{\epsilon}$ and $\omega < \omega_p/\sqrt{\epsilon}$ respectively.

-
- [1] J. Xia, Y. Maeno, P. T. Beyersdorf, M. M. Fejer, and A. Kapitulnik, Phys. Rev. Lett. **97**, 167002 (2006).
 - [2] G. M. Luke *et al.*, Nature **394**, 558 (1998).
 - [3] M. Sigrist and K. Ueda, Rev. Mod. Phys. **63**, 239 (1991).
 - [4] A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).
 - [5] A. P. Mackenzie and Y. Maeno, Rev. Mod. Phys. **75**, 657 (2003).
 - [6] F. Kidwingira, J. D. Strand, D. J. Van Harlingen, and Y. Maeno, Science **314**, 1267 (2006).
 - [7] S. Das Sarma, M. Freedman, C. Nayak, S. H. Simon, and A. Stern, eprint arXiv: 0707.1889 (2007).
 - [8] V. P. Mineev, Phys. Rev. B **76**, 212501 (2007).
 - [9] V. M. Yakovenko, Phys. Rev. Lett. **98**, 087003 (2007).
 - [10] S. K. Yip and J. A. Sauls, J. Low Temp. Phys. **86**, 257 (1992).
 - [11] Q. P. Li and R. Joynt, Phys. Rev. B **44**, 4720 (1991).
 - [12] G. E. Volovik, Sov. Phys. JETP **67**, 1804 (1988).
 - [13] A. Furusaki, M. Matsumoto, and M. Sigrist, Phys. Rev. B **64**, 54514 (2001).
 - [14] K. Sengupta, R. Roy, and M. Maiti, Phys. Rev. B **74**, 94505 (2006).
 - [15] G. E. Volovik and V. M. Yakovenko, J. Phys.: Condens. Matter **1**, 5263 (1989).
 - [16] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
 - [17] T. Senthil, J. B. Marston, and M. P. A. Fisher, Phys. Rev. B **60**, 4245 (1999).
 - [18] J. Goryo and K. Ishikawa, Phys. Lett. A **260**, 294 (1999).
 - [19] M. Stone and R. Roy, Phys. Rev. B **69**, 184511 (2004).
 - [20] T. Kita, J Phys. Soc. Jpn. **65**, 664 (1996).
 - [21] N. D. Mermin and P. Muzikar, Phys. Rev. B **21**, 980 (1980).
 - [22] J. R. Kirtley *et al.*, Phys. Rev. B **76**, 014526 (2007).
 - [23] P. G. Björnsson, Y. Maeno, M. E. Huber, and K. A. Moler, Phys. Rev. B **72**, 012504 (2005).
 - [24] J. Goryo and K. Ishikawa, Phys. Lett. A **246**, 549 (1998).
 - [25] B. Horowitz and A. Golub, Europhys. Lett **57**, 892 (2002).
 - [26] P. I. Arseev, S. O. Loiko, and N. K. Fedorov, Sov. Phys. Usp. **49**, 1 (2006).
 - [27] V. Ambegaokar and L. P. Kadanoff, Nuovo Cimento B **10**, 914 (1961).
 - [28] A. J. Millis, Phys. Rev. B **35**, 151 (1987).
 - [29] T. Katsufuji, M. Kasai, and Y. Tokura, Phys. Rev. Lett. **76**, 126 (1996).

- [30] H. Ehrenreich and M. H. Cohen, Phys. Rev. **115**, 786 (1959).
- [31] R. M. White and T. H. Geballe, *Long range order in solids* (Academic Press New York, 1979).
- [32] L. D. Landau and E. M. Lifshitz, *Course of theoretical physics. vol. 8: Electrodynamics of continuous media*. (Oxford).
- [33] T. Oguchi, Phys. Rev. B **51**, 1385 (1995).
- [34] R. M. Lutchyn, P. Nagornykh, and V. M. Yakovenko, arXiv:0801.4175 (2008).
- [35] More generally, unlike in a non-chiral superconductor, this coupling persists when the Hamiltonian has a three fold or higher rotation symmetry.
- [36] We note that a neutral chiral superfluid would likewise have a transverse component to the mass current, since the term $\frac{\tilde{c}_{xy}\omega}{e}(\hat{z} \times \mathbf{k})(\Phi/2)$ persists in this case, though the term $\tilde{c}_{xy}(\hat{z} \times \nabla A_0)$ would be absent.
- [37] The condition of Galilean invariance can be relaxed to that of threefold or higher rotational symmetry as before.
- [38] This value is obtained by using $\epsilon = 10$ and $\omega_p = 2.88eV$ (based on optical data in the normal state [29]) in $(n + i\kappa)^2 = \epsilon - \omega_p^2/\omega^2$. It is desirable to obtain optical data in the superconducting state.
- [39] We have used $k_{\parallel} \approx \frac{2\pi}{l}$, where $l = 25\mu m$ is the beam diameter and $k_f/m = 5.5 \times 10^4 ms^{-1}$ corresponding to the gamma band [5].
- [40] We thank Roman Lutchyn for a useful discussion on this point. Also see Ref. [34].